

GROTHENDIEK'S HOMOTOPY HYPOTHESIS

AMRANI ILIAS

ABSTRACT. We construct a "diagonal" cofibrantly generated model structure on the category of simplicial objects in the category of topological categories $\mathbf{sCat}_{\mathbf{Top}}$, which is the category of diagrams $[\Delta^{op}, \mathbf{Cat}_{\mathbf{Top}}]$. Moreover, we prove that the diagonal model structure is left proper and cellular. We also prove that the category of ∞ -groupoids (the full subcategory of topological categories) has a cofibrantly generated model structure and is Quillen equivalent to the model category of simplicial sets, which proves the Grothendieck's homotopy hypothesis.

INTRODUCTION AND RESULTS

This article can be seen as a first application of the existence of a model structure on the category of small topological categories $\mathbf{Cat}_{\mathbf{Top}}$ [1], namely for proving the *Grothendieck's homotopy hypothesis*. Before talking about homotopy hypothesis, we describe our first result related to the algebraic \mathcal{K} -theory. In [9], Waldhausen defined the \mathcal{K} -theory of a Waldhausen category \mathbf{W} as homotopy groups of some grouper-like E_∞ -space $\mathcal{K}(\mathbf{W})$. He defined a sort of suspension which takes Waldhausen category \mathbf{W} to a simplicial Waldhausen category $\mathcal{S}_\bullet \mathbf{W}$. This category can be considered as a simplicial object in the category of small (topological) categories. The algebraic \mathcal{K} -theory of a suspension $\mathcal{K}(\mathcal{S}_\bullet \mathbf{W})$ is defined as the realization of the nerve taken degree-wise, more precisely $\mathcal{K}(\mathcal{S}_\bullet \mathbf{W}) = \text{diag } N_\bullet w\mathcal{S}_\bullet \mathbf{W}$. What is important here is the interpretation of $N_\bullet w\mathbf{D}$ for a given category \mathbf{D} . Indeed, it is the coherent nerve of the (topological) Dwyer-Kan localization of $w\mathbf{D}$ with respect to $w\mathbf{D}$, i.e., the coherent nerve of the ∞ -groupoid $L_{w\mathbf{D}} w\mathbf{D} := w\mathbf{D}[w\mathbf{D}^{-1}]$. More precisely, we have a weak equivalence $N_\bullet w\mathbf{D} \rightarrow \widetilde{N}_\bullet L_{w\mathbf{D}} w\mathbf{D}$ (under some good conditions). In fact, for each topological category \mathbf{A} we can associate its underlying ∞ -groupoid denoted by \mathbf{A}' . Our idea is to construct a model structure on $\mathbf{sCat}_{\mathbf{Top}}$ 1.2 having the following property: $\mathbf{A}_\bullet \rightarrow \mathbf{B}_\bullet$ is a weak equivalence if and only if $\text{diag } \widetilde{N}_\bullet \mathbf{A}'_\bullet \rightarrow \text{diag } \widetilde{N}_\bullet \mathbf{B}'_\bullet$ is a weak equivalence of simplicial sets. In [1], we have proved that there is a weak equivalence $k^! \widetilde{N}_\bullet \mathbf{A}'_\bullet \rightarrow \widetilde{N}_\bullet \mathbf{A}_\bullet$. It means that the left Quillen endofunctor $k^!$ capture the homotopy type of the underlying ∞ -groupoid associated to any topological category. Now, we can state our first result as follow

Theorem A.1.2 *There is a cofibrantly generated model structure on $\mathbf{sCat}_{\mathbf{Top}}$ (diagonal model structure) such that $\mathbf{A}_\bullet \rightarrow \mathbf{B}_\bullet$ is a weak equivalence (fibration) if and only if*

$$\text{diag } k^! \widetilde{N}_\bullet \mathbf{A}_\bullet \rightarrow \text{diag } k^! \widetilde{N}_\bullet \mathbf{B}_\bullet$$

Date: October 26, 2011.

2000 Mathematics Subject Classification. Primary 55U40, 55P10. Secondary 18F20, 18D25.

Key words and phrases. Model Categories, ∞ -groupoids.

is a weak equivalence (fibration) in \mathbf{sSet} . Or equivalently,

$$\mathrm{diag} \widetilde{N}_\bullet A'_\bullet \rightarrow \mathrm{diag} \widetilde{N}_\bullet B'_\bullet$$

is a weak equivalence in \mathbf{sSet} .

In the first section 1, we construct a new model structure on $\mathbf{sCat}_{\mathbf{Top}}$. In 1.9, we explain why it is harder to prove the existence of such diagonal model structure on $\mathbf{sCat}_{\mathbf{sSet}}$. In sections 2 and 3, we prove in details the left properness and the cellularity of the new model structure on $\mathbf{sCat}_{\mathbf{Top}}$.

Theorem B.2.8 *The new model structure on $\mathbf{sCat}_{\mathbf{Top}}$ is left proper.*

Theorem C.3.4 *The new model structure on $\mathbf{sCat}_{\mathbf{Top}}$ is cellular.*

Our goal was to construct the stable model category $\mathbf{Sp}^\Sigma(\mathbf{sCat}_{\mathbf{Top}}, S)$ of symmetric spectra over $\mathbf{sCat}_{\mathbf{Top}}$, with respect to some left quillen endofunctor S (suspension). Unfortunately the category $\mathbf{sCat}_{\mathbf{Top}}$ is not simplicial model category, but we believe that combining some technics from [6] and [3] we can give an equivalent model for $\mathbf{Sp}^\Sigma(\mathbf{sCat}_{\mathbf{Top}}, S)$.

Section 4, is quite independent from the previous sections. We prove that the category of topological categories which are also ∞ -groupoids is a model category.

Theorem D.4.4 *There exists cofibrantly generated model structure on the category of ∞ -groupoids (definition 4), where the weak equivalences are given by Dwyer-Kan equivalences.*

Finally, we prove the ultimate theorem related to the *Grothendieck's homotopy hypothesis*

Theorem (Grothendieck's homotopy hypothesis). 4.6 *The category of infinity groupoids is Quillen equivalent to the category of simplicial sets.*

1. MODEL STRUCTURE

We will use the same notations as in [1]

Notation 1.1.

- (1) We denote \mathbf{Top} the category of compactly generated Hausdorff spaces.
- (2) \mathbf{sSet}_K denotes the model category on simplicial sets where the fibrant object are Kan complexes. \mathbf{sSet}_Q denotes the Joyal model structure where the fibrant objects are quasi-categories (∞ -categories).
- (3) The functor $k_! : \mathbf{sSet}_K \rightarrow \mathbf{sSet}_Q$ is defined as the left Kan extension of the functor which takes Δ^n to the nerve of the groupoid with n objects and only one isomorphism between each 2 objects. Moreover $k_!$ has a right adjoint denoted by $k^!$.
- (4) The composition of functors

$$\mathbf{sSet}_K \xrightarrow{k_!} \mathbf{sSet}_Q \xrightarrow{\Xi} \mathbf{Cat}_{\mathbf{sSet}} \xrightarrow{|-|} \mathbf{Cat}_{\mathbf{Top}}$$

is denoted by $\Theta : \mathbf{sSet} \rightarrow \mathbf{Cat}_{\mathbf{Top}}$. The composition $\Xi \circ k_!$ is denoted by $\tilde{\Theta}$.

- (5) The composition

$$\mathbf{Cat}_{\mathbf{Top}} \xrightarrow{\mathrm{sing}} \mathbf{Cat}_{\mathbf{sSet}} \xrightarrow{\widetilde{N}_\bullet} \mathbf{sSet}_Q \xrightarrow{k^!} \mathbf{sSet}_K$$

is denoted by $\Psi : \mathbf{Cat}_{\mathbf{Top}} \rightarrow \mathbf{sSet}$. The composition $k^! \circ \widetilde{N}_\bullet$ is denoted by $\widetilde{\Psi}$.

- (6) \mathbf{sSet}_d^2 denotes the category of bisimplicial sets provided with the diagonal model structure called also *Moerdijk model structure*. There is a Quillen equivalence:

$$\mathbf{sSet}_{\mathbf{K}} \begin{array}{c} \xrightarrow{d_*} \\ \xleftarrow{\text{diag}} \end{array} \mathbf{sSet}_d^2$$

- (7) \mathbf{sSet}_{pr}^2 denotes the category of bisimplicial sets provided with the projective model structure. It is well known that every projective weak equivalence is a diagonal equivalence.
- (8) The category $\mathbf{Cat}_{\mathbf{sSet}}$ is equipped with Bergner model structure [2], $\mathbf{Cat}_{\mathbf{Top}}$ is equipped with the model structure defined in [1]. The functors $k_!$ [7], $|-|$ and Ξ are left Quillen functors. The functors $k^!$ [7], sing and \widetilde{N}_\bullet are right Quillen functors. Moreover, the adjunctions $(\Xi, \widetilde{N}_\bullet)$ and $(|-|, \text{sing})$ are Quillen equivalences [8], [1].
- (9) All objects in $\mathbf{Cat}_{\mathbf{Top}}$ are **fibrant**. The functor sing applied to a topological category is a fibrant simplicial category.

We should remind that (Θ, Ψ) (resp. $(\widetilde{\Theta}, \widetilde{\Psi})$) is a Quillen adjunction because it is a composition of Quillen adjunctions [1]. This adjoint pair is naturally extended to an adjunction between \mathbf{sSet}^2 and $\mathbf{sCat}_{\mathbf{Top}}$ (resp. $\mathbf{sCat}_{\mathbf{sSet}}$) denoted by $\Theta_\bullet, \Psi_\bullet$ (resp. $\widetilde{\Theta}_\bullet, \widetilde{\Psi}_\bullet$). Finally, we define the following adjunction:

$$\mathbf{sSet} \begin{array}{c} \xrightarrow{d_*} \\ \xleftarrow{\text{diag}} \end{array} \mathbf{sSet}^2 \begin{array}{c} \xrightarrow{\Theta_\bullet} \\ \xleftarrow{\Psi_\bullet} \end{array} \mathbf{sCat}_{\mathbf{Top}}$$

Now, we can state the main theorem for this section:

Theorem 1.2 (A). *The adjunction $(\Theta_\bullet, d_*, \text{diag} \Psi_\bullet)$ induces a cofibrantly generated model structure on $\mathbf{sCat}_{\mathbf{Top}}$, where*

- (1) *a morphism $f : \mathbf{C}_\bullet \rightarrow \mathbf{D}_\bullet$ in $\mathbf{sCat}_{\mathbf{Top}}$ is a weak equivalence (fibration) if $\text{diag} \Psi_\bullet f : \text{diag} \Psi_\bullet \mathbf{C}_\bullet \rightarrow \text{diag} \Psi_\bullet \mathbf{D}_\bullet$ is a weak equivalence (fibration) in $\mathbf{sSet}_{\mathbf{K}}$,*
- (2) *The generating acyclic cofibrations are given by $\Theta_\bullet d_* \Lambda_i^n \rightarrow \Theta_\bullet d_* \Delta^n$, for all $0 \leq n$ and $0 \leq i \leq n$,*
- (3) *The generating cofibrations are given by $\Theta_\bullet d_* \partial \Delta^n \rightarrow \Theta_\bullet d_* \Delta^n$, for all $0 \leq n$.*

We start with a useful lemma which gives us conditions to transfer a model structure by adjunction.

Lemma 1.3. [10], proposition 3.4.1] *Consider an adjunction*

$$\mathbf{M} \begin{array}{c} \xrightarrow{G} \\ \xleftarrow{F} \end{array} \mathbf{C}$$

where \mathbf{M} is a cofibrantly generated model category, with generating cofibrations I and generating trivial cofibrations J . We pose

- *W the class of morphisms in \mathbf{C} such the image by F is a weak equivalence in \mathbf{M} .*

- F the class of morphisms in \mathbf{C} such the image by F is a fibration in \mathbf{M} .

We suppose that the following conditions are verified:

- (1) The domains of $G(i)$ are small with respect to $G(I)$ for all $i \in I$ and the domains of $G(j)$ are small with respect to $G(J)$ for all $j \in J$.
- (2) The functor F commutes with directed colimits i.e.,

$$F\text{colim}(\lambda \rightarrow \mathbf{C}) = \text{colim}F(\lambda \rightarrow \mathbf{C}).$$

- (3) Every transfinite composition of weak equivalences in \mathbf{M} is a weak equivalence.
- (4) The pushout of $G(j)$ by any morphism f in \mathbf{C} is in W .

Then \mathbf{C} forms a model category with weak equivalences (resp. fibrations) W (resp. F). Moreover, it is cofibrantly generated with generating cofibrations $G(I)$ and generating trivial cofibrations $G(J)$.

In order to prove the main theorem 1.2 we follow the lemma 1.3.

Lemma 1.4. *Let A a simplicial subset of B such that the inclusion $A \rightarrow B$ is a weak equivalence. Let \mathbf{C} an object in $\mathbf{Cat}_{\mathbf{Top}}$. Then for all $F \in \mathbf{hom}_{\mathbf{Cat}_{\mathbf{Top}}}(\Theta(A), \mathbf{C})$ the functor Ψ sends the following pushout*

$$\begin{array}{ccc} \Theta(A) & \xrightarrow{F} & \mathbf{C} \\ \downarrow & & \downarrow \\ \Theta(B) & \longrightarrow & \mathbf{D} \end{array}$$

to a homotopy cocartesian square in \mathbf{sSet} .

Proof. Since Θ is a quillen functor, $\Theta(A) \rightarrow \Theta(B)$ is a trivial cofibration in $\mathbf{Cat}_{\mathbf{Top}}$. It implies that $\mathbf{C} \rightarrow \mathbf{D}$ is an equivalence in $\mathbf{Cat}_{\mathbf{Top}}$, and so $\text{sing}\mathbf{C} \rightarrow \text{sing}\mathbf{D}$ is an equivalence between fibrant objects in $\mathbf{Cat}_{\mathbf{sSet}}$. It follows that $\widetilde{N}_{\bullet}\text{sing}\mathbf{C} \rightarrow \widetilde{N}_{\bullet}\text{sing}\mathbf{D}$ is an equivalence between fibrant objects (quasi-category) in $\mathbf{sSet}_{\mathbf{Q}}$. Finally, $k^!\widetilde{N}_{\bullet}\text{sing}\mathbf{C} \rightarrow k^!\widetilde{N}_{\bullet}\text{sing}\mathbf{D}$ i.e., $\Psi\mathbf{C} \rightarrow \Psi\mathbf{D}$ is an equivalence in $\mathbf{sSet}_{\mathbf{K}}$. By the same argument, $\Psi\Theta(A) \rightarrow \Psi\Theta(B)$ is a weak equivalence in $\mathbf{sSet}_{\mathbf{K}}$. Moreover, it is a monomorphism since $\Theta(A) \rightarrow \Theta(B)$ admits a section (all objects in $\mathbf{Cat}_{\mathbf{Top}}$ are fibrant). So $\Psi\Theta(A) \rightarrow \Psi\Theta(B)$ is a trivial cofibration in $\mathbf{sSet}_{\mathbf{K}}$, consequently $\Psi\mathbf{C} \rightarrow \Psi\Theta(B) \sqcup_{\Psi\Theta(A)} \Psi\mathbf{C}$ is an equivalence in $\mathbf{sSet}_{\mathbf{K}}$. The following diagram summarize the situation:

$$\begin{array}{ccc} \Psi\Theta(A) & \longrightarrow & \Psi\mathbf{C} \\ \downarrow \sim & & \downarrow \sim \\ \Psi\Theta(B) & \longrightarrow & \Psi\Theta(B) \sqcup_{\Psi\Theta(A)} \Psi\mathbf{C} \\ & \searrow & \downarrow t \\ & & \Psi\mathbf{D} \end{array}$$

(Note: A curved arrow labeled \sim also connects $\Psi\mathbf{C}$ to $\Psi\mathbf{D}$.)

we conclude that $t : \Psi\Theta(B) \sqcup_{\Psi\Theta(A)} \Psi\mathbf{C} \rightarrow \Psi\mathbf{D}$ is a weak equivalence in $\mathbf{sSet}_{\mathbf{K}}$ \square

More generally, we consider the following bisimplicial sets (cf [4])

$$B = d_*\Delta^n = \bigsqcup_{\beta \in \Delta^n} \Delta^n.$$

$$A = d_* \Lambda_i^n = \bigsqcup_{\beta \in \Lambda_i^n} C^\beta, \text{ where } C^\beta \text{ are weakly contractible.}$$

$$S = d_* \partial \Delta^n = \bigsqcup_{\beta \in \partial \Delta^n} D^\beta, \text{ where } D^\beta \text{ are weakly contractible.}$$

Lemma 1.5. *If $i : S \rightarrow B$ is a generating cofibration in \mathbf{sSet}_d^2 (resp. an acyclic generating cofibration $j : A \rightarrow B$ in \mathbf{sSet}_d^2) and \mathbf{C}_\bullet an object of $\mathbf{sCat}_{\mathbf{Top}}$, then the functor Ψ_\bullet sends the following pushouts*

$$\begin{array}{ccc} \Theta_\bullet(S) & \longrightarrow & \mathbf{C}_\bullet \\ \downarrow & & \downarrow \\ \Theta_\bullet(B) & \longrightarrow & \mathbf{D}_\bullet \end{array} \quad \begin{array}{ccc} \Theta_\bullet(A) & \longrightarrow & \mathbf{C}_\bullet \\ \downarrow & & \downarrow \\ \Theta_\bullet(B) & \longrightarrow & \mathbf{D}_\bullet \end{array}$$

to homotopy cocartesian squares in \mathbf{sSet}_{pr}^2 .

Proof. We will do the proof for $i : S \rightarrow B$, the other case is analogue. We denote by $\Delta^n(m)$ (resp. $\partial \Delta^n(m)$) the set of m -simplices Δ^n (resp. $\partial \Delta^n$). First of all, let remark that

$$j_m : S_m = \bigsqcup_{\beta \in \partial \Delta^n(m)} D^\beta \rightarrow \bigsqcup_{\beta \in \partial \Delta^n(m)} \Delta^n = B'_m$$

is a trivial cofibration in $\mathbf{sSet}_{\mathbf{K}}$. In an other hand, colimits in $\mathbf{sCat}_{\mathbf{Top}}$ are computed degree-wise. In degree m we have that

$$\mathbf{D}_m = (\mathbf{C}_m \bigsqcup_{\Theta S_m} \Theta B'_m) \bigsqcup_{\beta \in (\Delta^n(m) \setminus \partial \Delta^n(m))} \Theta(\Delta^n)$$

If we consider now the pushout in \mathbf{sSet}^2

$$\begin{array}{ccc} \Psi_\bullet \Theta_\bullet(S) & \longrightarrow & \Psi_\bullet \mathbf{C}_\bullet \\ \downarrow & & \downarrow \\ \Psi_\bullet \Theta_\bullet(B) & \longrightarrow & X \end{array}$$

then X_m is equal to

$$(\Psi \mathbf{C}_m \bigsqcup_{\Psi \Theta S_m} \Psi \Theta B'_m) \bigsqcup_{\beta \in (\Delta^n(m) \setminus \partial \Delta^n(m))} \Psi \Theta(\Delta^n).$$

By the lemma 1.4, the map $\Psi \mathbf{C}_m \bigsqcup_{\Psi \Theta S_m} \Psi \Theta B'_m \rightarrow \Psi(\mathbf{C}_m \bigsqcup_{\Theta S_m} \Theta B'_m)$ is a weak equivalence in $\mathbf{sSet}_{\mathbf{K}}$. Consequently, $X_m \rightarrow \Psi \mathbf{D}_m$, is an equivalence for each m . So $X \rightarrow \Psi_\bullet \mathbf{D}_\bullet$ is a weak equivalence in \mathbf{sSet}_{pr}^2 . It follows that a diagonal weak equivalence in \mathbf{sSet}_d^2 \square

Lemma 1.6. *Let $A \rightarrow B$ be an acyclic cofibration in \mathbf{sSet}_d^2 , then the induced morphisms in \mathbf{sSet} , $\text{diag} \Psi_\bullet \Theta_\bullet(A) \rightarrow \text{diag} \Psi_\bullet \Theta_\bullet(B)$, is an acyclic cofibration in $\mathbf{sSet}_{\mathbf{K}}$.*

Proof. If $Y \rightarrow *$ is an equivalence in $\mathbf{sSet}_{\mathbf{K}}$, then $\Theta(Y) \rightarrow *$ is an equivalence in $\mathbf{Cat}_{\mathbf{Top}}$ since Θ is a left Quillen functor. We have the following commutative diagram:

$$\begin{array}{ccc} \Theta_{\bullet} A & \xrightarrow{f} & \bigsqcup_{\Delta^n} * \\ \downarrow & & \downarrow \\ \Theta_{\bullet} B & \xrightarrow{g} & \bigsqcup_{\Delta^n} * \end{array}$$

where f, g are equivalences of topological categories degree by degree. Applying the functor Ψ_{\bullet} we have a degree-wise equivalence of bisimplicial sets \mathbf{sSet}_{pr}^2 , because all objects in $\mathbf{Cat}_{\mathbf{Top}}$ are fibrant. Now, applying the diagonal functor, we conclude that $\text{diag} \Psi_{\bullet} \Theta_{\bullet}(A) \rightarrow \text{diag} \Psi_{\bullet} \Theta_{\bullet}(B)$ is an equivalence. To see that is in fact a cofibration of simplicial sets, it is enough to see that $\Theta(C^{\beta}) \rightarrow \Theta(\Delta^n)$ is a trivial cofibration of topological categories, consequently, it admits a section because all objects in $\mathbf{Cat}_{\mathbf{Top}}$ are fibrant. This implies that $\Psi_{\bullet} \Theta_{\bullet}(A) \rightarrow \Psi_{\bullet} \Theta_{\bullet}(B)$ is a degree-wise monomorphism of bisimplicial sets. Finally, applying the functor diag we obtain that

$$\text{diag} \Psi_{\bullet} \Theta_{\bullet}(A) \rightarrow \text{diag} \Psi_{\bullet} \Theta_{\bullet}(B)$$

is a monomorphism in \mathbf{sSet} . □

Corollary 1.7. *With the same notations as in lemma 1.5, the map of bisimplicial sets $\Psi_{\bullet} \mathbf{C}_{\bullet} \rightarrow X$ is a diagonal weak equivalence. Moreover the map $\Psi_{\bullet} \mathbf{C}_{\bullet} \rightarrow \Psi_{\bullet} \mathbf{D}_{\bullet}$ is a weak diagonal equivalence.*

Proof. Since the functor diag commutes with colimits, lemmas 1.5 and 1.6 imply that $\text{diag} \Psi_{\bullet} \mathbf{C}_{\bullet} \rightarrow \text{diag} X$ is a weak equivalence. By the lemma 1.5 we have that $\text{diag} X \rightarrow \text{diag} \Psi_{\bullet} \mathbf{D}_{\bullet}$ is a weak equivalence. So the property 2 out of 3 the map $\Psi_{\bullet} \mathbf{C}_{\bullet} \rightarrow \Psi_{\bullet} \mathbf{D}_{\bullet}$ is a diagonal equivalence. □

Lemma 1.8. *The functors $k^!$, $\widetilde{\mathbf{N}}_{\bullet}$ and sing commute with directed colimits.*

Proof. The fact $k^!$ commutes with directed colimits is a direct consequence from the adjunction $(k_!, k^!)$ and that the functor $\mathbf{hom}_{\mathbf{sSet}}(k_! \Delta^n, -)$ commutes with directed colimits. By the same way $\widetilde{\mathbf{N}}_{\bullet}$ commutes with directed colimits since $\Xi(\Delta^n)$ are small object in $\mathbf{Cat}_{\mathbf{sSet}}$. The functor $\text{sing} : \mathbf{Cat}_{\mathbf{Top}} \rightarrow \mathbf{Cat}_{\mathbf{sSet}}$ commutes with directed colimits by [1]. □

Finally, we are ready to prove the main theorem of this section

Proof of the main theorem 1.2. First of all, $\mathbf{sCat}_{\mathbf{Top}}$ is complete and cocomplete because $\mathbf{Cat}_{\mathbf{Top}}$ is so. Following the fundamental lemma 1.3, the points (1) and (3) are obvious. the point (2) is proven in 1.8 and finally, the point (4) is given by 1.7. □

Remark 1.9. We should point out that we are unable to prove a same result for $\mathbf{sCat}_{\mathbf{sSet}}$ for the simple reason that objects in $\mathbf{Cat}_{\mathbf{sSet}}$ are not all fibrant. As we have seen before, it plays a crucial role to prove the main theorem 1.2. However, we believe that such model structure exists and is Quillen equivalent to the diagonal model structure on $\mathbf{sCat}_{\mathbf{Top}}$. The main idea is to prove that given any simplicial category \mathbf{C} , the counite map $k^! \widetilde{\mathbf{N}}_{\bullet} \mathbf{C} \rightarrow k^! \widetilde{\mathbf{N}}_{\bullet} \text{sing}[\mathbf{C}]$ is a weak equivalence in $\mathbf{sSet}_{\mathbf{K}}$,

this statement is true if \mathbf{C} was fibrant.

2. LEFT PROPERNESS

In this section we will show that $\mathbf{sCat}_{\mathbf{Top}}$ is a left proper model category. First of all, we will give some properties of cofibrations.

Lemma 2.1. *Let $i : A \rightarrow B$ be a generating cofibration in \mathbf{sSet}_d^2 , then $\Theta_\bullet i : \Theta_\bullet A \rightarrow \Theta_\bullet B$ is an inclusion of topological categories. Moreover, $\Psi_\bullet \Theta_\bullet i : \Psi_\bullet \Theta_\bullet A \rightarrow \Psi_\bullet \Theta_\bullet B$ is a monomorphism in \mathbf{sSet}^2 .*

Proof. We have seen in 1.5 that the map $i_m : A_m \rightarrow B_m$ is written as

$$i_m : A_m \rightarrow B'_m \bigsqcup_{\beta \in \Delta^n(m) \setminus \partial \Delta^n(m)} \Delta^n.$$

The corestriction map $i'_m : A_m \rightarrow B'_m$ is a trivial cofibration in \mathbf{sSet}_K . So, $\Theta i'_m : \Theta A_m \rightarrow \Theta B'_m$ is a trivial cofibration in $\mathbf{Cat}_{\mathbf{Top}}$, consequently, we have a section for i'_m because all objects in $\mathbf{Cat}_{\mathbf{Top}}$ are fibrant. We conclude that i_m is an inclusion of topological categories and Ψi_m is a monomorphism in \mathbf{sSet} . \square

Lemma 2.2. *Let $\mathbf{A}_\bullet \rightarrow \mathbf{B}_\bullet$ be a cellular cofibration obtained by a pushout in $\mathbf{sCat}_{\mathbf{Top}}$ of a generating cofibration $\Theta_\bullet i : \Theta_\bullet Z \rightarrow \Theta_\bullet W$. Then $\mathbf{A}_\bullet \rightarrow \mathbf{B}_\bullet$ is a degree-wise inclusion of topological categories. Moreover, $\Psi_\bullet \mathbf{A}_\bullet \rightarrow \Psi_\bullet \mathbf{B}_\bullet$ is a monomorphism in \mathbf{sSet}^2 .*

Proof. First of all,

$$\mathbf{B}_m = (\mathbf{A}_m \bigsqcup_{\Theta Z_m} \Theta W'_m) \bigsqcup_{\beta \in (\Delta^n(m) \setminus \partial \Delta^n(m))} \Theta(\Delta^n),$$

where the corestriction

$$\Theta_\bullet i'_m : \mathbf{A}_m \rightarrow \mathbf{A}_m \bigsqcup_{\Theta Z_m} \Theta W'_m$$

is a trivial cofibration between fibrant objects in $\mathbf{Cat}_{\mathbf{Top}}$. This imply that $\Theta_\bullet i'_m$ admits a section; it follows that $\Theta_\bullet i_m$ is a degree-wise inclusion of topological categories and

$$\Psi_\bullet \mathbf{A}_\bullet \rightarrow \Psi_\bullet \mathbf{B}_\bullet$$

is a monomorphism in \mathbf{sSet}^2 . \square

Corollary 2.3. *Let $i : \mathbf{A}_\bullet \rightarrow \mathbf{B}_\bullet$ be a cofibration in $\mathbf{sCat}_{\mathbf{Top}}$, then i_m is an inclusion of topological categories and $\Psi_\bullet i$ is a degree-wise monomorphism in \mathbf{sSet}^2 .*

Proof. For the case of cellular cofibrations, it is a direct consequence of 2.2. We know that monomorphisms are colesed under retracts. We conclude that cofibrations in $\mathbf{sCat}_{\mathbf{Top}}$ are degree-wise inclusions. On an other hand, $\Psi = k^! \widetilde{\mathbf{N}}_\bullet \text{sing}$ preserves inclusions, it follows that $\Psi_\bullet i$ is a monomorphism in \mathbf{sSet}^2 . \square

Lemma 2.4. *Let $\mathbf{A}_\bullet \rightarrow \mathbf{B}_\bullet$ be a cofibration obtained by pushout from a generating cofibration $\Theta_\bullet(A) \rightarrow \Theta_\bullet(B)$ in $\mathbf{sCat}_{\mathbf{Top}}$. Then the functor Ψ_\bullet sends the following pushout to*

$$\begin{array}{ccc} \mathbf{A}_\bullet & \longrightarrow & \mathbf{C}_\bullet \\ \downarrow & & \downarrow \\ \mathbf{B}_\bullet & \longrightarrow & \mathbf{D}_\bullet \end{array}$$

to a homotopy cocartesian square in \mathbf{sSet}_{pr}^2 . More generally, let $\mathbf{A}_\bullet \rightarrow \mathbf{B}_\bullet$ a cellular cofibration in $\mathbf{sCat}_{\mathbf{Top}}$, then we have the same conclusion.

Proof. By the same arguments as in 1.5, we have

$$\mathbf{B}_m = (\mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m) \bigsqcup_{\beta \in (\Delta^n(m) \setminus \partial \Delta^n(m))} \Theta(\Delta^n)$$

with the property that $\mathbf{A}_m \rightarrow \mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m$ is trivial cofibration in $\mathbf{Cat}_{\mathbf{Top}}$, it follows that it admits a section. Consequently $\Psi \mathbf{A}_m \rightarrow \Psi(\mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m)$ is a trivial cofibration in \mathbf{sSet} . On the other hand,

$$\mathbf{D}_m = \mathbf{C}_m \bigsqcup_{\mathbf{A}_m} \mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m \bigsqcup_{\beta \in (\Delta^n(m) \setminus \partial \Delta^n(m))} \Theta(\Delta^n);$$

applying the functor Ψ , we have the universal map in \mathbf{sSet} given by

$$\Psi \mathbf{C}_m \bigsqcup_{\Psi \mathbf{A}_m} \Psi \mathbf{B}_m \rightarrow \Psi(\mathbf{C}_m \bigsqcup_{\mathbf{A}_m} \mathbf{B}_m).$$

Since $\mathbf{A}_\bullet \rightarrow \mathbf{B}_\bullet$ is obtained as a pushout of a generating cofibration in $\mathbf{sCat}_{\mathbf{Top}}$, we have

$$\Psi \mathbf{C}_m \bigsqcup_{\Psi \mathbf{A}_m} \Psi \mathbf{B}_m = \Psi \mathbf{C}_m \bigsqcup_{\Psi \mathbf{A}_m} \Psi(\mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m) \bigsqcup_{\beta \in (\Delta^n(m) \setminus \partial \Delta^n(m))} \Psi \Theta(\Delta^n)$$

Since $\Psi \mathbf{A}_m \rightarrow \Psi(\mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m)$ is a trivial cofibration in $\mathbf{sSet}_{\mathbf{K}}$, then

$$\Psi \mathbf{C}_m \rightarrow \Psi \mathbf{C}_m \bigsqcup_{\Psi \mathbf{A}_m} \Psi(\mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m)$$

is also a trivial cofibration. On the other hand,

$$\Psi \mathbf{C}_m \rightarrow \Psi(\mathbf{C}_m \bigsqcup_{\Theta A_m} \Theta B'_m) = \Psi(\mathbf{C}_m \bigsqcup_{\mathbf{A}_m} \mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m)$$

is an equivalence by 1.4. We have the commutative diagram:

$$\begin{array}{ccc} \Psi \mathbf{C}_m & \xrightarrow{\sim} & \Psi \mathbf{C}_m \bigsqcup_{\Psi \mathbf{A}_m} \Psi(\mathbf{A}_m \bigsqcup_{\Theta A_m} \Theta B'_m) \\ & \searrow \sim & \swarrow \sim \\ & \Psi(\mathbf{C}_m \bigsqcup_{\Theta A_m} \Theta B'_m) & \end{array}$$

It follows that

$$\begin{array}{c} \Psi C_m \sqcup_{\Psi A_m} \Psi(A_m \sqcup_{\Theta A_m} \Theta B'_m) \sqcup \sqcup_{\beta \in (\Delta^n(m) \setminus \partial \Delta^n(m))} \Psi \Theta(\Delta^n) \\ \downarrow \\ \Psi(C_m \sqcup_{\Theta A_m} \Theta B'_m) \sqcup \sqcup_{\beta \in (\Delta^n(m) \setminus \partial \Delta^n(m))} \Psi \Theta(\Delta^n) \end{array}$$

is a weak equivalence in \mathbf{sSet}_K . consequently

$$\Psi C_m \sqcup_{\Psi A_m} \Psi B_m \rightarrow \Psi D_n$$

is a weak equivalence.

For the general case of cellular cofibrations, we write $i : A_\bullet \rightarrow B_\bullet$ as a transfinite composition

$$A_\bullet \rightarrow A_\bullet^1 \rightarrow \dots A_\bullet^\alpha \rightarrow A_\bullet^{\alpha+1} \rightarrow \dots \rightarrow B_\bullet.$$

We pose $C_\bullet^\alpha = C_\bullet \sqcup_{A_\bullet} A_\bullet^\alpha$, then the morphism $C_\bullet \rightarrow D_\bullet$ is a transfinite composition

$$C_\bullet \rightarrow C_\bullet^1 \rightarrow \dots C_\bullet^\alpha \rightarrow C_\bullet^{\alpha+1} \rightarrow \dots \rightarrow D_\bullet.$$

By the precedent case:

$$\Psi_\bullet A_\bullet^\alpha \sqcup_{\Psi_\bullet A_\bullet} \Psi_\bullet C_\bullet \rightarrow \Psi_\bullet C_\bullet^\alpha$$

is a degree-wise weak equivalence. Moreover, $\Psi_\bullet A_\bullet^\alpha \rightarrow \Psi_\bullet A_\bullet^{\alpha+1}$ is a monomorphism in \mathbf{sSet}^2 by 2.3. we conclude that:

$$\operatorname{colim}_\alpha \Psi_\bullet A_\bullet^\alpha \sqcup_{\Psi_\bullet A_\bullet} \Psi_\bullet C_\bullet \rightarrow \operatorname{colim}_\alpha \Psi_\bullet C_\bullet^\alpha$$

is a weak equivalence. Noting that Ψ_\bullet commutes with directed colimits, we conclude that

$$\Psi_\bullet B_\bullet \sqcup_{\Psi_\bullet A_\bullet} \Psi_\bullet C_\bullet \rightarrow \Psi_\bullet D_\bullet.$$

is a degree-wise weak equivalence and so a diagonal equivalence. □

Corollary 2.5. *Let $i : A_\bullet \rightarrow B_\bullet$ as in 2.4, the the pushout in $\mathbf{sCat}_{\mathbf{Top}}$ of a weak equivalence along i is a weak equivalence.*

Proof. We note the pushout diagram by:

$$\begin{array}{ccc} A_\bullet & \xrightarrow{\sim} & C_\bullet \\ \downarrow & & \downarrow \\ B_\bullet & \longrightarrow & D_\bullet. \end{array}$$

applying the functor $\operatorname{diag} \Psi_\bullet$ to the diagram, we obtain a homotopy cocartesian diagram in \mathbf{sSet}_{pr}^2 . By lemma 2.3, the morphism $\Psi_\bullet A_\bullet \rightarrow \Psi_\bullet B_\bullet$ is a monomorphism in \mathbf{sSet}^2 , consequently $\operatorname{diag} \Psi_\bullet A_\bullet \rightarrow \operatorname{diag} \Psi_\bullet B_\bullet$ is a cofibration in \mathbf{sSet} . The

following pushout diagram in \mathbf{sSet} summarize the situation:

$$\begin{array}{ccc}
 \text{diag}\Psi_{\bullet}\mathbf{A}_{\bullet} & \xrightarrow{\sim} & \text{diag}\Psi_{\bullet}\mathbf{C}_{\bullet} \\
 \downarrow & & \downarrow \\
 \text{diag}\Psi_{\bullet}\mathbf{B}_{\bullet} & \xrightarrow{f} & X \\
 & \searrow t & \nearrow g \\
 & & \text{diag}\Psi_{\bullet}\mathbf{D}_{\bullet}
 \end{array}$$

Since \mathbf{sSet} is left proper, f is a weak equivalence. Moreover, g is an a weak equivalence by 2.4. consequently, t is a weak equivalence. \square

Corollary 2.6. *If $i : \mathbf{A}_{\bullet} \rightarrow \mathbf{B}_{\bullet}$ is a cellular cofibration in $\mathbf{sCat}_{\mathbf{Top}}$, then the pushout of a weak equivalence along i is again a weak equivalence.*

Proof. Consider the following pushout :

$$\begin{array}{ccc}
 \mathbf{A}_{\bullet} & \xrightarrow{\sim} & \mathbf{C}_{\bullet} \\
 \downarrow & & \downarrow \\
 \mathbf{B}_{\bullet} & \longrightarrow & \mathbf{D}_{\bullet}
 \end{array}$$

We write $i : \mathbf{A}_{\bullet} \rightarrow \mathbf{B}_{\bullet}$ as a transfinite composition of morphisms as described in corollary 2.5 i.e.,

$$\mathbf{A}_{\bullet} \rightarrow \mathbf{A}_{\bullet}^1 \rightarrow \dots \mathbf{A}_{\bullet}^{\alpha} \rightarrow \mathbf{A}_{\bullet}^{\alpha+1} \rightarrow \dots \rightarrow \mathbf{B}_{\bullet}.$$

If we pose $\mathbf{C}_{\bullet}^{\alpha} = \mathbf{C}_{\bullet} \sqcup_{\mathbf{A}_{\bullet}} \mathbf{A}_{\bullet}^{\alpha}$, then the morphism $\mathbf{C}_{\bullet} \rightarrow \mathbf{D}_{\bullet}$ is the transfinite composition

$$\mathbf{C}_{\bullet} \rightarrow \mathbf{C}_{\bullet}^1 \rightarrow \dots \mathbf{C}_{\bullet}^{\alpha} \rightarrow \mathbf{C}_{\bullet}^{\alpha+1} \rightarrow \dots \rightarrow \mathbf{D}_{\bullet}.$$

By corollary 2.5 $\text{diag}\Psi_{\bullet}\mathbf{A}_{\bullet}^{\alpha} \rightarrow \text{diag}\Psi_{\bullet}\mathbf{C}_{\bullet}^{\alpha}$ is a weak equivalence in $\mathbf{sSet}_{\mathbf{K}}$. We conclude that

$$\mathbf{B}_{\bullet} = \text{colim}_{\alpha} \mathbf{A}_{\bullet}^{\alpha} \rightarrow \text{colim}_{\alpha} \mathbf{C}_{\bullet}^{\alpha} = \mathbf{D}_{\bullet}$$

is a weak equivalence in $\mathbf{sCat}_{\mathbf{Top}}$. \square

Lemma 2.7. *If $i' : \mathbf{A}'_{\bullet} \rightarrow \mathbf{B}'_{\bullet}$ is a retract of a cellular cofibration in $\mathbf{sCat}_{\mathbf{Top}}$, then the pushout of a weak equivalence along i' is again a weak equivalence.*

Proof. By hypothesis, $i' : \mathbf{A}'_{\bullet} \rightarrow \mathbf{B}'_{\bullet}$ is a retract of some cellular cofibration $i : \mathbf{A}_{\bullet} \rightarrow \mathbf{B}_{\bullet}$. Let the following pushout diagram in $\mathbf{sCat}_{\mathbf{Top}}$

$$\begin{array}{ccc}
 \mathbf{A}'_{\bullet} & \xrightarrow{\sim} & \mathbf{C}_{\bullet} \\
 \downarrow i' & & \downarrow j' \\
 \mathbf{B}'_{\bullet} & \longrightarrow & \mathbf{B}'_{\bullet} \sqcup_{\mathbf{A}'_{\bullet}} \mathbf{C}_{\bullet}
 \end{array}$$

The retraction between i and i' induces a retraction between $\mathbf{C}_{\bullet} \rightarrow \mathbf{B}'_{\bullet} \sqcup_{\mathbf{A}'_{\bullet}} \mathbf{C}_{\bullet} = \mathbf{D}'_{\bullet}$ and $\mathbf{C}_{\bullet} \rightarrow \mathbf{B}_{\bullet} \sqcup_{\mathbf{A}_{\bullet}} \mathbf{C}_{\bullet} = \mathbf{D}_{\bullet}$. Consequently,

$$t' : \Psi_{\bullet}\mathbf{B}'_{\bullet} \sqcup_{\Psi_{\bullet}\mathbf{A}'_{\bullet}} \Psi_{\bullet}\mathbf{C}_{\bullet} \rightarrow \Psi_{\bullet}\mathbf{D}'_{\bullet}$$

is a retract of

$$t : \Psi_{\bullet} \mathbf{B}_{\bullet} \bigsqcup_{\Psi_{\bullet} \mathbf{A}_{\bullet}} \Psi_{\bullet} \mathbf{C}_{\bullet} \rightarrow \Psi_{\bullet} \mathbf{D}_{\bullet}.$$

By lemma 2.4, the map t is a weak equivalence and so t' a weak equivalence (by retract). The map $\text{diag} \Psi_{\bullet} \mathbf{A}'_{\bullet} \rightarrow \text{diag} \Psi_{\bullet} \mathbf{B}'_{\bullet}$ is a cofibration in \mathbf{sSet} by lemma 2.3, so

$$\Psi_{\bullet} \mathbf{B}'_{\bullet} \rightarrow \Psi_{\bullet} \mathbf{B}'_{\bullet} \bigsqcup_{\Psi_{\bullet} \mathbf{A}'_{\bullet}} \Psi_{\bullet} \mathbf{C}_{\bullet}$$

is an weak equivalence (diagonal) in \mathbf{sSet}_d^2 since \mathbf{sSet} is left proper. Consequently,

$$\Psi_{\bullet} \mathbf{B}'_{\bullet} \rightarrow \Psi_{\bullet} \mathbf{D}'_{\bullet}$$

is a diagonal equivalence since t' is degree-wise equivalence. \square

Theorem 2.8. *[B] The model category $\mathbf{sCat}_{\mathbf{Top}}$ is left proper.*

Proof. It is a direct consequence from 2.7. \square

3. CELLULARITY OF $\mathbf{sCat}_{\mathbf{Top}}$

In this section, we prove that $\mathbf{sCat}_{\mathbf{Top}}$ is a cellular model category (cf [5]).

Lemma 3.1. *The domains and codomains of generating cofibration of the diagonal model structure on $\mathbf{sCat}_{\mathbf{Top}}$ are compact.*

Proof. Suppose that $\mathbf{C}_{\bullet} \rightarrow \mathbf{D}_{\bullet}$ is a cellular cofibration $\mathbf{sCat}_{\mathbf{Top}}$. Let $\mathbf{A}_{\bullet} \rightarrow \mathbf{D}_{\bullet}$ be a morphism where $\mathbf{A}_{\bullet} = \Theta_{\bullet} d_* X$ is a (co)domain of some generating cofibration $\mathbf{sCat}_{\mathbf{Top}}$. The map $\mathbf{C}_{\bullet} \rightarrow \mathbf{D}_{\bullet}$ is written as transfinite composition

$$\mathbf{C}_{\bullet} = \mathbf{C}_{\bullet}^0 \rightarrow \mathbf{C}_{\bullet}^1 \dots \mathbf{C}_{\bullet}^s \rightarrow \dots \mathbf{D}_{\bullet}.$$

Applying the functor $\text{diag} \Psi_{\bullet}$ to this diagram, we obtain:

$$\text{diag} \Psi_{\bullet} \mathbf{C}_{\bullet} = \text{diag} \Psi_{\bullet} \mathbf{C}_{\bullet}^0 \rightarrow \text{diag} \Psi_{\bullet} \mathbf{C}_{\bullet}^1 \dots \rightarrow \text{diag} \Psi_{\bullet} \mathbf{C}_{\bullet}^s \rightarrow \dots \text{diag} \Psi_{\bullet} \mathbf{D}_{\bullet}.$$

But $\text{diag} \Psi_{\bullet} \mathbf{C}_{\bullet}^s \rightarrow \text{diag} \Psi_{\bullet} \mathbf{C}_{\bullet}^{s+1}$ is a cofibration in \mathbf{sSet} by 2.1. By adjunction, a map $\mathbf{A}_{\bullet} \rightarrow \mathbf{D}_{\bullet}$ is the same thing as giving a map f in \mathbf{sSet} $f : X \rightarrow \text{diag} \Psi_{\bullet} \mathbf{D}_{\bullet}$. Since X is compact in \mathbf{sSet} , this imply that f is factored for a certain s by $g : X \rightarrow \text{diag} \Psi_{\bullet} \mathbf{C}_{\bullet}^s$. Using the adjunction again, we conclude that $\mathbf{A}_{\bullet} \rightarrow \mathbf{D}_{\bullet}$ is factored by $\Theta_{\bullet} d_* X \rightarrow \mathbf{C}_{\bullet}^s$. \square

Lemma 3.2. *The domains of generating acyclic cofibration in $\mathbf{sCat}_{\mathbf{Top}}$ are small relatively to the cellular cofibration.*

Proof. We use the same notations as in lemma 3.1. Let $\text{colim}_s \mathbf{C}_{\bullet}^s$, such that $\mathbf{C}_{\bullet}^i \rightarrow \mathbf{C}_{\bullet}^{i+1}$ be a directed colimit which is a cellular cofibration. The goal is to prove that $\mathbf{hom}_{\mathbf{sCat}_{\mathbf{Top}}}(\mathbf{A}_{\bullet}, -)$ commutes with directed colimits, where $\mathbf{A}_{\bullet} = \Theta_{\bullet} d_* X$ is a domain of an acyclic cofibration in $\mathbf{sCat}_{\mathbf{Top}}$. Again, by adjunction we have,

$$\mathbf{hom}_{\mathbf{sCat}_{\mathbf{Top}}}(\mathbf{A}_{\bullet}, \text{colim}_s \mathbf{C}_{\bullet}^s) = \mathbf{hom}_{\mathbf{sSet}}(X, \text{diag} \Phi_{\bullet} \text{colim}_s \mathbf{C}_{\bullet}^s).$$

But $\text{diag} \Phi_{\bullet}$ commutes with directed colimits, so

$$\mathbf{hom}_{\mathbf{sSet}}(X, \text{diag} \Phi_{\bullet} \text{colim}_s \mathbf{C}_{\bullet}^s) = \mathbf{hom}_{\mathbf{sSet}}(X, \text{colim}_s \text{diag} \Phi_{\bullet} \mathbf{C}_{\bullet}^s).$$

But all objects in \mathbf{sSet} are small. Consequently:

$$\mathbf{hom}_{\mathbf{sSet}}(X, \text{colim}_s \text{diag} \Phi_{\bullet} \mathbf{C}_{\bullet}^s) = \text{colim}_s \mathbf{hom}_{\mathbf{sSet}}(X, \text{diag} \Phi_{\bullet} \mathbf{C}_{\bullet}^s).$$

Finally, we conclude by adjunction that

$$\mathbf{hom}_{\mathbf{sCat}_{\mathbf{Top}}}(\mathbf{A}_\bullet, \operatorname{colim}_s \mathbf{C}_\bullet^s) = \operatorname{colim}_s \mathbf{hom}_{\mathbf{sCat}_{\mathbf{Top}}}(\mathbf{A}_\bullet, \mathbf{C}_\bullet^s).$$

□

Lemma 3.3. *The cofibration in $\mathbf{sCat}_{\mathbf{Top}}$ are effective monomorphisms.*

Proof. Let $\mathbf{C}_\bullet \xrightarrow{i} \mathbf{D}_\bullet$ be any cofibration in $\mathbf{sCat}_{\mathbf{Top}}$ (in particular it is an inclusion of categories). The goal is to compute the equalizer of the following diagram:

$$\mathbf{D}_\bullet \rightrightarrows \mathbf{D}_\bullet \sqcup_{\mathbf{C}_\bullet} \mathbf{D}_\bullet$$

where the two maps are inclusions of categories coming from the following pushout diagram:

$$\begin{array}{ccc} \mathbf{C}_\bullet & \xrightarrow{i} & \mathbf{D}_\bullet \\ \downarrow i & & \downarrow i_1 \\ \mathbf{D}_\bullet & \xrightarrow{i_2} & \mathbf{D}_\bullet \sqcup_{\mathbf{C}_\bullet} \mathbf{D}_\bullet \end{array}$$

We claim that the equalizer is given exactly by

$$\mathbf{C}_\bullet \xrightarrow{i} \mathbf{D}_\bullet \rightrightarrows \mathbf{D}_\bullet \sqcup_{\mathbf{C}_\bullet} \mathbf{D}_\bullet$$

First of all, we remark that is a commutative diagram. Suppose that \mathbf{C}'_\bullet is an other candidate for the equalizer. Since the functor $\operatorname{Ob} : \mathbf{sCat} \rightarrow \mathbf{sSet}$ commutes with (co)limits (Ob admits a left and a right adjoint), There exists a unique map t which makes the following diagram be commutative:

$$\begin{array}{ccc} \operatorname{Ob} \mathbf{C}'_\bullet & & \\ \downarrow t & \searrow \operatorname{Ob}(F) & \\ \operatorname{Ob} \mathbf{C}_\bullet & \xrightarrow{\operatorname{Ob}(i)} & \operatorname{Ob} \mathbf{D}_\bullet \rightrightarrows \operatorname{Ob} \mathbf{D}_\bullet \sqcup_{\operatorname{Ob} \mathbf{C}_\bullet} \operatorname{Ob} \mathbf{D}_\bullet \end{array}$$

Indeed, the cofibrations in $\mathbf{sCat}_{\mathbf{Top}}$ are injective at the level of objects 2.3, and \mathbf{sSet} is cellular [5]. Now, let γ be a morphism in \mathbf{C}'_\bullet such that $i_1 F(\gamma) = i_2 F(\gamma)$. Since $i_1 : \mathbf{C}_\bullet \rightarrow \mathbf{D}_\bullet \sqcup_{\mathbf{C}_\bullet} \mathbf{D}_\bullet$ and $i_2 : \mathbf{C}_\bullet \rightarrow \mathbf{D}_\bullet \sqcup_{\mathbf{C}_\bullet} \mathbf{D}_\bullet$ are inclusions of categories, this implies that $F(\gamma)$ is a morphism in \mathbf{C}_\bullet . We conclude that any morphism $F : \mathbf{C}'_\bullet \rightarrow \mathbf{D}_\bullet$ in $\mathbf{sCat}_{\mathbf{Top}}$ such that $i_1 F = i_2 F$ is uniquely factored as a composition:

$$\mathbf{C}'_\bullet \rightarrow \mathbf{C}_\bullet \rightarrow \mathbf{D}_\bullet.$$

□

Corollary 3.4. *The model category $\mathbf{sCat}_{\mathbf{Top}}$ is cellular.*

4. MODEL STRUCTURE ON THE CATEGORY OF ∞ -GROUPOIDS

In this section we will prove the existence of a natural cofibrantly model structure on the categories of ∞ -groupoids.

Definition 4.1. Let \mathbf{C} be a topological category, we will say that \mathbf{C} is an ∞ -groupoid if $\pi_0 \mathbf{C}$ (the associated homotopy category) is a groupoid.

For every topological category \mathbf{D} we can associate its underlying ∞ -groupoid $G\mathbf{D}$ by the following pullback diagram:

$$\begin{array}{ccc} G\mathbf{D} = \text{iso } \pi_0 \mathbf{C} \times_{\pi_0 \mathbf{D}} \mathbf{D} & \longrightarrow & \mathbf{D} \\ \downarrow & & \downarrow \\ \text{iso } \pi_0 \mathbf{D} & \xrightarrow{\sim} & \pi_0 \mathbf{D}. \end{array}$$

Notation 4.2. The category of small ∞ -groupoids will be denoted by $\infty - \mathbf{Grp}$.

Lemma 4.3. *Let $f : \mathbf{C} \rightarrow \mathbf{D}$ be a map of ∞ -groupoids, then f is a Dwyer-Kan equivalence of topological categories [1] if and only if Ψf is a weak equivalence in \mathbf{sSet}_K .*

Proof. Suppose that f is a Dwyer-Kan equivalence. We know that Ψ is a right Quillen functor and all objects in $\mathbf{Cat}_{\mathbf{Top}}$ are fibrant. We conclude that Ψf is a weak equivalence in \mathbf{sSet}_K . Conversely, suppose that Ψf ($k^! \widetilde{N}_\bullet \text{sing} f$) is a weak equivalence in \mathbf{sSet}_K , we remark that $\Psi \mathbf{C}, \widetilde{N}_\bullet \text{sing} \mathbf{C}$, $\Psi \mathbf{D}$ and $\widetilde{N}_\bullet \text{sing} \mathbf{D}$ are Kan complexes since \mathbf{C} and \mathbf{D} are ∞ -groupoids [[1], section 6]. We have the following commutative diagram of weak equivalences [[1], section 6]:

$$\begin{array}{ccc} \Psi \mathbf{C} & \xrightarrow{\sim} & \Psi \mathbf{D} \\ \downarrow \sim & & \downarrow \sim \\ J \widetilde{N}_\bullet \text{sing} \mathbf{C} & \xrightarrow{\sim} & J \widetilde{N}_\bullet \text{sing} \mathbf{D} \\ \downarrow id & & \downarrow id \\ \widetilde{N}_\bullet \text{sing} \mathbf{C} & \xrightarrow{\sim} & \widetilde{N}_\bullet \text{sing} \mathbf{D} \end{array}$$

where J is the Joyal endofunctor on \mathbf{sSet} (more precisely the subcategory of quasi-categories) [7] which associate to each ∞ -category the biggest Kan sub complex. Moreover the maps $\Psi \mathbf{C} \rightarrow \widetilde{N}_\bullet \text{sing} \mathbf{C}$ and $\Psi \mathbf{D} \rightarrow \widetilde{N}_\bullet \text{sing} \mathbf{D}$ are trivial fibrations in \mathbf{sSet}_K . But \mathbf{sSet}_K is a left Bousfield localization [[7], proposition 6.15] of \mathbf{sSet}_Q , it means that $\widetilde{N}_\bullet \text{sing} \mathbf{C} \rightarrow \widetilde{N}_\bullet \text{sing} \mathbf{D}$ is an equivalence of ∞ -categories and so we conclude that $\text{sing} \mathbf{C} \rightarrow \text{sing} \mathbf{D}$ is a Dwyer-Kan equivalence of simplicial categories, consequently $\mathbf{C} \rightarrow \mathbf{D}$ is a Dwyer-Kan equivalence of topological categories. \square

Theorem 4.4 (D). *The adjunction (Θ, Ψ) induces a cofibrantly generated model structure on $\infty - \mathbf{Grp}$, where*

- (1) *a morphism $f : \mathbf{C} \rightarrow \mathbf{D}$ in $\infty - \text{groupoids}$ is a weak equivalence (fibration) if*

$$\Psi f : \Psi \mathbf{C} \rightarrow \Psi \mathbf{D}$$

is a weak equivalence (fibration) in \mathbf{sSet}_K ,

- (2) *The generating acyclic cofibrations are given by $\Theta \Delta_i^n \rightarrow \Theta \Delta^n$, for all $0 \leq n$ and $0 \leq i \leq n$,*
- (3) *The generating cofibrations are given by $\Theta \partial \Delta^n \rightarrow \Theta \Delta^n$, for all $0 \leq n$.*

Proof. The category $\infty - \mathbf{Grp}$ is complete by definition and cocomplete because the functor $\pi_0 : \mathbf{Cat}_{\mathbf{Top}} \rightarrow \mathbf{Cat}$ commutes with colimits (has a right adjoint) and the category \mathbf{Grp} (classical groupoids) is cocomplete. Moreover Θ takes any simplicial

set to an ∞ -groupoid since it commutes with colimits and $\Theta(\Delta^n)$ is obviously an ∞ -groupoid. Following lemma 1.3, we have to check only the condition 4. Let us take a generating acyclic cofibration $\Theta\Lambda_i^n \rightarrow \Theta\Delta^n$, we know that is a Dwyer-Kan equivalence and a cofibration of topological categories since Θ is a left Quillen functor. If we consider the following pushout in $\infty - \mathbf{Grp}$:

$$\begin{array}{ccc} \Theta\Lambda_i^n & \longrightarrow & \mathbf{C} \\ \downarrow \sim & & \downarrow f \\ \Theta\Delta^n & \longrightarrow & \mathbf{D} \end{array}$$

we can deduce that f is a Dwyer-Kan equivalence of topological categories since $\mathbf{Cat}_{\mathbf{Top}}$ has the appropriate model structure [1]. Finally, we conclude by lemma 4.3 that $\Psi\mathbf{C} \rightarrow \Psi\mathbf{D}$ is a weak equivalence in $\mathbf{sSet}_{\mathbf{K}}$. \square

Remark 4.5. We don't know if the category $\infty - \mathbf{Grp}$ is left proper, but it is right proper for obvious reasons.

Theorem 4.6 (Grothendieck homotopy hypothesis). *The Quillen adjunction*

$$\mathbf{sSet}_{\mathbf{K}} \xrightleftharpoons[\Psi]{\Theta} \infty - \mathbf{Grp}$$

induces a Quillen equivalence.

Proof. We should mention the we can't prove the statement directly i.e., that the unit and the counit are equivalences. First we prove that the functor $\widetilde{N}_{\bullet} \text{sing} : \infty - \mathbf{Grp} \rightarrow \mathbf{sSet}_{\mathbf{K}}$ is well defined. Let \mathbf{C} be an infinity groupoid then we know [1] that $\text{sing}\mathbf{C}$ is a simplicial (fibrant) infinity groupoid and that $\widetilde{N}_{\bullet} \text{sing}\mathbf{C}$ is a Kan complex. Consequently the functor $\widetilde{N}_{\bullet} \text{sing}$ takes Dwyer-Kan equivalences (fibrations) to equivalences (fibrations) in $\mathbf{sSet}_{\mathbf{K}}$ (since $\mathbf{sSet}_{\mathbf{K}}$ is left Bousfield localization of $\mathbf{sSet}_{\mathbf{Q}}$). So the functor $\widetilde{N}_{\bullet} \text{sing}$ is a well defined right Quillen functor. On the other hand, let \mathbf{C} and \mathbf{D} in $\infty - \mathbf{Grp}$, we have the following commutative diagram of isomorphisms of (derived) mapping spaces in $\text{Ho}(\mathbf{sSet}_{\mathbf{K}})$:

$$\begin{array}{ccc} \mathbf{map}_{\mathbf{Cat}_{\mathbf{Top}}}(\mathbf{C}, \mathbf{D}) & \xrightarrow{\sim} & \mathbf{map}_{\mathbf{sSet}_{\mathbf{Q}}}(\widetilde{N}_{\bullet} \text{sing}\mathbf{C}, \widetilde{N}_{\bullet} \text{sing}\mathbf{D}) \\ \downarrow = & & \downarrow = \\ \mathbf{map}_{\infty - \mathbf{Grp}}(\mathbf{C}, \mathbf{D}) & \xrightarrow{f} & \mathbf{map}_{\mathbf{sSet}_{\mathbf{K}}}(\widetilde{N}_{\bullet} \text{sing}\mathbf{C}, \widetilde{N}_{\bullet} \text{sing}\mathbf{D}) \end{array}$$

The first isomorphism

$$\mathbf{map}_{\mathbf{Cat}_{\mathbf{Top}}}(\mathbf{C}, \mathbf{D}) \rightarrow \mathbf{map}_{\mathbf{sSet}_{\mathbf{Q}}}(\widetilde{N}_{\bullet} \text{sing}\mathbf{C}, \widetilde{N}_{\bullet} \text{sing}\mathbf{D})$$

comes from the fact that $\widetilde{N}_{\bullet} \text{sing}$ is a Quillen equivalence [2], [8], [1]. The first equality $\mathbf{map}_{\mathbf{sSet}_{\mathbf{K}}}(\widetilde{N}_{\bullet} \text{sing}\mathbf{C}, \widetilde{N}_{\bullet} \text{sing}\mathbf{D}) = \mathbf{map}_{\mathbf{sSet}_{\mathbf{Q}}}(\widetilde{N}_{\bullet} \text{sing}\mathbf{C}, \widetilde{N}_{\bullet} \text{sing}\mathbf{D})$ is a consequence of the fact that $\widetilde{N}_{\bullet} \text{sing}\mathbf{D}$ is a Kan complex. The second equality $\mathbf{map}_{\mathbf{Cat}_{\mathbf{Top}}}(\mathbf{C}, \mathbf{D}) = \mathbf{map}_{\infty - \mathbf{Grp}}(\mathbf{C}, \mathbf{D})$ is a consequence of the fact that the model full subcategory $\infty - \mathbf{Grp}$ of $\mathbf{Cat}_{\mathbf{Top}}$ has the same weak equivalences (Dwyer-Kan equivalences 4.3) and \mathbf{C} and \mathbf{D} are infinity groupoids. We conclude that

$$\widetilde{N}_{\bullet} \text{sing} : \text{Ho}(\infty - \mathbf{Grp}) \rightarrow \text{Ho}(\mathbf{sSet}_{\mathbf{K}})$$

is fully faithful. Now we prove that $\widetilde{N}_\bullet \text{sing}$ is essentially surjective. Recall from [7] that for any simplicial set X the natural transformation $\nu_X : X \rightarrow k_! X$ is a weak equivalence in \mathbf{sSet}_K , so that the map:

$$X \rightarrow k_!(X) \rightarrow \widetilde{N}_\bullet \text{sing} |\Xi(k_!(X))|$$

is a weak equivalence in \mathbf{sSet}_K since the second map is the unit map of the adjunction (Quillen equivalence) between $\mathbf{Cat}_{\mathbf{Top}}$ and \mathbf{sSet}_Q which is a weak equivalence of quasi-categories and so a weak equivalence in \mathbf{sSet}_K . But $|\Xi(k_!(X))|$ is an infinity groupoid. We conclude that $\widetilde{N}_\bullet \text{sing}$ is essentially surjective. On the other hand, for any infinity groupoid \mathbf{C} we have that $k^! \widetilde{N}_\bullet \text{sing} \mathbf{C} \rightarrow J \widetilde{N}_\bullet \text{sing} \mathbf{C} = \widetilde{N}_\bullet \text{sing} \mathbf{C}$ is a trivial fibration [7],[1]. Consequently, the functor

$$k^! \widetilde{N}_\bullet \text{sing} : \text{Ho}(\infty - \mathbf{Grp}) \rightarrow \text{Ho}(\mathbf{sSet}_K)$$

is an equivalence of homotopical (ordinary) categories and its left adjoint is exactly $|\Xi(k_!(-))|$. Finally, we conclude that the adjunction (Θ, Ψ) induces a Quillen equivalence between $\infty - \mathbf{Grp}$ and \mathbf{sSet}_K . \square

Remark 4.7. The diagonal model structure on $\mathbf{sCat}_{\mathbf{Top}}$ can be *restricted* to a diagonal model structure on $[\Delta^{op}, \infty - \mathbf{Grp}]$. We are pretty sure that this new model structure is also equivalent to \mathbf{sSet}_K . Moreover, it is cellular and left proper.

4.1. $n - \mathbf{Groupoids}$. It is well known that any connected topological space X is (zigzag) equivalent to $\mathcal{B}Y$ where Y is a topological monoid group like equivalent to ΩX . We explain the same result using homotopy hypothesis i.e., every topological space is zig-zag equivalent to a topological space of the form

$$\bigsqcup_{x \in \pi_0(X)} \mathcal{B}A_x$$

where A_x is a homotopical group (strict multiplication and the inverses are defined up to homotopy) and \mathcal{B} is the bar construction.

In order to explain this phenomenon, we should recall the interpretation of the coherent nerve $\widetilde{N}_\bullet \text{sing}$ for a topological groupoid \mathbf{C} . For simplicity we take \mathbf{C} with one object x and suppose that $\text{End}_{\mathbf{C}}(x, x)$ is a topological group (in general it is a homotopical group), then the geometric realization of $\widetilde{N}_\bullet \text{sing} \mathbf{C}$ is nothing else than a model for $\mathcal{B}\text{End}_{\mathbf{C}}(x, x)$ the Bar construction of $\text{End}_{\mathbf{C}}(x, x)$ i.e.,

$$|\widetilde{N}_\bullet \text{sing} \mathbf{C}| \sim \mathcal{B}\text{End}_{\mathbf{C}}(x, x).$$

In general, if X is a topological space we associate the ∞ -groupoid $\mathbf{G}(X)$ given by the formula 4.4

$$X \mapsto \mathbf{G}(X) = |\Xi k_! \text{sing}(X)|.$$

By Grothendieck homotopy hypothesis theorem 4.6, we know that the unit map is an equivalence and the map $\text{sing}(X) \rightarrow k_! \text{sing}(X)$ is also an equivalence [7]

$$\text{sing}(X) \xrightarrow{\sim} k_! \text{sing}(X) \xrightarrow{\sim} \widetilde{N}_\bullet \text{sing} \mathbf{G}(X)$$

is an equivalence. On the other hand, the topological realization of the coherent nerve $\widetilde{N}_\bullet \text{sing} \mathbf{G}(X)$ is equivalent to

$$\bigsqcup_{x \in [\mathbf{G}(X)]} \mathcal{B} \text{End}_{\mathbf{G}(X)}(x, x)$$

where $[\mathbf{G}(X)]$ is the set of chosen objects x of $\mathbf{G}(X)$, one in each connected component of $\mathbf{G}(X)$. finally, we end-up with the following zig-zag equivalence

$$X \xleftarrow{\sim} |\mathrm{sing}(X)| \xrightarrow{\sim} |\widetilde{\mathbf{N}}_{\bullet} \mathrm{sing} \mathbf{G}(X)| \xleftarrow{\sim} \bigsqcup_{x \in [\mathbf{G}(X)]} \mathcal{B} \mathrm{End}_{\mathbf{G}(X)}(x, x).$$

Definition 4.8. The category $\mathbf{n} - \mathbf{Type}$ is the full subcategory of \mathbf{Top} consisting of spaces with the property that all homotopy groups greater than n are vanishing. We say that an ∞ -groupoid is an n -groupoid if it is enriched over topological spaces of type $n - 1$. We denote the category of n -groupoids by $\mathbf{n} - \mathbf{Grp}$.

Remark 4.9. We conclude that the homotopy category $\mathrm{Ho}(\mathbf{n} - \mathbf{Type}) \subset \mathrm{Ho}(\mathbf{Top})$ of spaces of type n is equivalent to the homotopy category $\mathrm{Ho}(\mathbf{n} - \mathbf{Grp}) \subset \mathrm{Ho}(\infty - \mathbf{Grp})$ of n -groupoids.

REFERENCES

- [1] I. Amrani. Model structure on the category of topological categories. *arxiv.org/pdf/1110.2695*, 2011.
- [2] J.E. Bergner. A model category structure on the category of simplicial categories. *Transactions-American Mathematical Society*, 359(5):2043, 2007.
- [3] D. Dugger. Replacing model categories with simplicial ones. *Transactions of the American Mathematical Society*, pages 5003–5027, 2001.
- [4] P.G. Goerss and JF Jardine. *Simplicial homotopy theory*. Birkhauser, 1999.
- [5] P. Hirschhorn. Model categories and their localizations. *Mathematical Surveys and Monographs*, page 99, 2002.
- [6] M. Hovey. Spectra and symmetric spectra in general model categories. *Journal of Pure and Applied Algebra*, 165(1):63–127, 2001.
- [7] A. Joyal. Advanced course on simplicial methods in higher categories. *CRM*, 2008.
- [8] J. Lurie. Higher topos theory. *Arxiv preprint math.CT/0608040*, 2006.
- [9] F. Waldhausen. Algebraic K-theory of spaces. *Algebraic and geometric topology (New Brunswick, NJ, 1983)*, 1126:318–419.
- [10] K. Worytkiewicz, K. Hess, P.E. Parent, and A. Tonks. A model structure a la Thomason on 2-Cat. *J. Pure Appl. Algebra*, 208(1):205–236, 2007.

DEPARTMENT OF MATHEMATICS, MASARYK UNIVERSITY, CZECH REPUBLIC

E-mail address: `ilias.amranifedotov@gmail.com`

E-mail address: `amrani@math.muni.cz`